Comparative Analysis of the Markowitz Model and the Single-Index Model: An Empirical Study on N/T Ratios and Investment Constraints

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Abstract. Two models have been extensively applied in the use of the modern portfolio theory. Markowitz model (MM), based on the mean–variance framework, prescribes precise theoretical implication of the efficient frontier under direct modeling of inter-asset covariance structure. Even though theoretically correct, however, MM does nevertheless continue to harbor serious shortcomings, viz., excessively high complication of computation and over-sensitiveness toward estimation error of the input data, particularly the estimation of the covariance matrix. Vehemently opposed to MM, on the other hand, is the Single-Index Model (IM) matrix, which incorporates the market index, a general factor, so drastically reducing the dimension of estimation of inter-asset correlation, and thus correspondingly increasing significantly the tractability and speed of computation. Unfortunately, however, the simplification does bring over-reliance toward the market factor, in effect excluding rigorous inter-asset correlations and thus potentially warping the shape of efficient frontier. Accordingly, both theoretically and practically, there will remain controversies over balancing accuracy and tractability between MM and IM. It undertakes systematic comparisons of the performance between the Markowitz model and the Single-Index Model under diverse conditions of varying sample sizes (T) of the data set, asset numbers (N), and constraint conditions of investment. It is discovered, under conditions of large-dimensional, small-sample, and no constraint, IM exhibits superior robustness, whereas under conditions of sufficiently large sample size and constraint, MM exhibits higher competitiveness. By filling the gap of theoretical and application values, the paper presents a concrete basis of model selection of application of portfolio optimization.

Keywords: Markowitz Model, Single-Index Model, portfolio optimization, out-of-sample performance, weight constrints.

1. Introduction

Modern Portfolio Theory (MPT), stablished the foundation for portfolio optimization through the mean-variance framework. The Markowitz Model (MM) explicitly employs the covariance matrix to capture correlations among assets, thereby providing a precise theoretical description of the efficient frontier. However, in high-dimensional and small-sample settings, the instability of covariance matrix estimation renders the MM highly vulnerable to estimation errors and prone to overfitting [1, 2].

Sharpe introduced the Single-Index Model (IM), which assumes that asset returns are driven by a common market factor, thereby substantially simplifying the covariance structure and greatly reducing the complexity of parameter estimation. Although the IM enhances model tractability and robustness, it may excessively simplify cross-sectional relationships among assets [3]. This trade-off between accuracy and parsimony remains a central issue of debate in both academic and practical domains.

In practice, portfolio construction often takes place under realistic constraints, such as long-only restrictions or concentration limits. These constraints act as implicit regularizers, mitigating extreme portfolio weights and stabilizing estimation noise [4, 5]. Recent studies highlight that constraints can fundamentally reshape model performance, further motivating a systematic comparison between MM and IM in constrained versus unconstrained settings.

Despite the extensive literature on portfolio optimization, most empirical studies focus on single aspects such as unconstrained optimization or specific market environments. Systematic comparisons across multiple dimensions-including sample length, asset universe size, and investment constraints-remain limited. Furthermore, the robustness of model choice under high-dimensional estimation risk has only recently been revisited with advances in covariance shrinkage and regularization [6, 7].

This study addresses these gaps by examining the relative performance of MM and IM under varying N/T ratios and portfolio constraints, with objectives of both risk minimization and return maximization. By integrating a multi-dimensional empirical design, the paper not only enriches the theoretical understanding of covariance- versus factor-based models but also provides practical guidance for portfolio managers operating under real-world constraints.

2. Methodology

2.1. Model Foundations

2.1.1 Markowitz model

The Markowitz model, rooted in Modern Portfolio Theory, formulates portfolio optimization under the mean–variance framework. Consider a portfolio of N risky assets with expected return vector $\mu = (\mu_1, \mu_2, \dots, \mu_N)^T$, covariance matrix $\Sigma \in R^{N\times N}$, and portfolio weights $w = (w_1, w_2, \dots, w_N)^T$, subject to $\sum_{i=1}^{N} w_i = 1$.

The expected portfolio return is:

$$\mu_p = w^T\mu \tag{1}$$

The portfolio variance, which measures total risk, is:

$$\sigma_p^2 = w^T\Sigma w \tag{2}$$

The standard Markowitz optimization problem is expressed as:

$$\min_{w} w^T \Sigma w \ \text{ s.t. } \ w^T\mu = \mu^*, \sum_{i=1}^{N} w_i = 1, w_i \geq 0, \ \forall I \tag{3}$$

Where μ^* is the target expected return. This formulation provides a theoretically precise characterization of the efficient frontier but requires estimation of Σ, involving $O(N^2)$ parameters, making it sensitive to estimation errors, especially in high-dimensional, small-sample settings.

2.1.2 Index model

To reduce the complexity of covariance estimation, Sharpe (1963) proposed the Single-Index Model, which assumes that the return of each asset is primarily driven by a common market factor. Specifically, for asset i:

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{4}$$

Where R_m is the return of the market index, β_i is the sensitivity of asset iii to the market factor, and ϵ_i is an idiosyncratic error term with $E[\epsilon_i] = 0$ and $\mathrm{Var}(\epsilon_i) = \sigma_{\epsilon_i}^2$.

The covariance between two assets i and j is then approximated as:

$$\mathrm{Cov}(R_i, R_j) = \beta_i\beta_j\sigma_m^2 \tag{5}$$

Where $\sigma_m^2 = \mathrm{Var}(R_m)$. The variance of asset iii is given by:

$$\mathrm{Var}(R_i) = \beta_i^2\sigma_m^2 + \sigma_{\epsilon_i}^2 \tag{6}$$

Thus, the covariance matrix under IM reduces to:

$$\Sigma = \beta\beta^T\sigma_m^2 + D \tag{7}$$

Where $\beta = (\beta_1, \beta_2, \dots, \beta_N)^T$, $D = \mathrm{diag}(\sigma_{\epsilon_1}^2, \dots, \sigma_{\epsilon_N}^2)$

This simplification reduces the number of parameters to estimate from $O(N^2)$ in MM to $O(N)$ in IM, thereby improving computational tractability and robustness. However, it does so at the expense of ignoring asset-specific correlations not explained by the market factor.

2.2. Dimension Design for Comparison

To comprehensively evaluate the performance of the Markowitz Model (MM) and the Single-Index Model (IM), this study designs the empirical analysis along three dimensions: sample length, number of assets, and investment constraints.

2.2.1 Sample length (T)

To investigate the effect of estimation window size, we consider rolling samples of 36, 60, and 120 months, corresponding to short-, medium-, and long-term horizons. This design allows us to capture the trade-off between responsiveness to new information and estimation stability.

2.2.2 Number of assets (N)

Portfolios are constructed with asset universe sizes of N=8, N=20, N=50. This setup reflects different levels of portfolio diversification and dimensionality, thereby enabling assessment of model robustness under varying cross-sectional scales.

2.2.3 Investment constraints

Three types of portfolio constraints are imposed to approximate realistic investment settings:

Unconstrained portfolios: weights are unrestricted apart from the budget condition $1^{\top}w = 1$.

Long-only with upper bound: all positions satisfy $wi \geq 0$ and individual asset weights are capped at 20%.

Budget-only portfolios: the sole restriction is the full investment constraint, $1^{\top}w = 1$, without short-sale or upper-bound limitations.

This three-dimensional design enables a systematic comparison of MM and IM, providing insights into how estimation length, portfolio size, and practical constraints jointly affect portfolio performance.

2.3. Evaluation Matrics

To rigorously compare the performance of the Markowitz Model (MM) and the Single-Index Model (IM), we employ a comprehensive set of out-of-sample evaluation metrics. These metrics capture portfolio efficiency, robustness, and risk characteristics under different experimental settings. The data sources and construction procedures are described in detail in Section 3.

2.3.1 Out-of-sample performance

We evaluate risk-adjusted performance by computing the annualized Sharpe ratio and information ratio, which are standard measures widely used in the literature for portfolio performance evaluation [8]. For portfolio excess return series $R_{p,t}$, the annualized Sharpe ratio is defined as

$$SR = \frac{E[R_{p,t}]}{\sqrt{Var(R_{p,t})}} \times \sqrt{12} \tag{8}$$

While the information ratio is computed as

$$IR = \frac{E[R_{p,t} - R_{b,t}]}{\sqrt{Var(R_{p,t} - R_{b,t})}} \times \sqrt{12} \tag{9}$$

Where $R_{b,t}$, denotes the benchmark return.

2.3.2 Forecasting accuracy and weight stability

To assess estimation reliability, we calculate the variance forecast error between realized and predicted portfolio variance. In addition, portfolio weight stability is measured using the Herfindahl index,

$$H = \sum_{i=1}^{N} w_{i,t}^{2} \tag{10}$$

Which reflects portfolio concentration and sensitivity to input perturbations.

2.3.3 Downside and tail risk

Extreme risk is evaluated through maximum drawdown (MDD) and expected shortfall (ES). Expected shortfall at confidence level α\alphaα is defined as

$$\mathrm{ES}_{\alpha} = \mathrm{E}\left[-\mathrm{R}_{\mathrm{p,t}} \mid \mathrm{R}_{\mathrm{p,t}} \leq \mathrm{q}_{\alpha}\right] \tag{11}$$

Where q_{α} is the α-quantile of the portfolio return distribution.

2.3.4 Statistical significance

To test whether performance differences between models are statistically significant, we conduct Newey-West adjusted t-tests, which account for heteroskedasticity and autocorrelation in return series.

By jointly considering performance, robustness, and statistical reliability, this multi-faceted evaluation framework ensures that the comparison between MM and IM is both comprehensive and empirically rigorous.

2.4. Data and Backtesting Setup

The empirical analysis is based on monthly return data obtained from Bloomberg, covering the period from September 2004 to September 2024. The dataset includes sector Exchange-Traded Funds (ETFs) as well as representative large-cap equities from the U.S. market. All return series are adjusted for dividends and corporate actions to ensure consistency. The risk-free rate is proxied by the one-month U.S. Treasury bill yield, also retrieved from Bloomberg.

To avoid look-ahead bias and approximate realistic investment practice, we implement a rolling-window estimation scheme. Specifically, for each rebalancing date, expected returns and covariance matrices (for MM) or factor loadings (for IM) are estimated using the most recent T months of data (T=36,60,120).

After estimating parameters, portfolios are constructed according to the specified optimization objectives (minimum variance or tangency) and subject to the given investment constraints. These portfolios are held for one month, and their realized returns are recorded. The process is then repeated by rolling the estimation window forward by one period. This iterative estimation–allocation–evaluation process produces out-of-sample return series, which are subsequently assessed using the performance metrics introduced in Section 2.3.

This setup ensures that the comparison between the Markowitz model and the Single-Index model is grounded in a consistent and forward-looking framework, with Bloomberg data providing a comprehensive and reliable empirical basis.

3. Empirical Results

3.1. Performance Under Different N/T Ratios

Figure 1 report the out-of-sample Sharpe ratio differences between the Markowitz Model (MM) and the Single-Index Model (IM) across varying values of N/T. The difference is defined as SRMM-SRIM, with positive values indicating superior performance of MM.

Under low N/T conditions (e.g., N=8 with T=36, 60, 120), MM exhibits a consistent but modest advantage over IM, with Sharpe improvements ranging from 0.14 to 0.17. This suggests that when the effective sample size is relatively large compared to the number of assets, MM can utilize its richer covariance structure to capture diversification benefits beyond what the single-factor approximation offers.

At moderate N/T ratios (e.g., N=20), the advantage of MM remains visible, though relatively smaller, with improvements around 0.13 to 0.16. In this regime, IM's parsimony provides robustness, while MM still gains from more accurate covariance estimation as the sample length grows.

In contrast, under high N/T conditions (e.g., N=50), MM substantially outperforms IM, with Sharpe differences increasing to 0.27, 0.29, and 0.44 for T=36, 60, 120, respectively. This pattern indicates that when the dimensionality burden is high, IM's simplified structure fails to capture the true correlation dynamics, whereas MM-especially when aided by Ledoit-Wolf shrinkage-delivers significantly superior out-of-sample efficiency.

Overall, the results confirm the theoretical trade-off: IM is relatively robust in high N/T environments, where small samples and high dimensionality limit covariance estimation accuracy. However, as N/T decreases with more available data or smaller portfolios, MM not only catches up but decisively dominates, underscoring the importance of aligning model choice with the effective information ratio in practice.

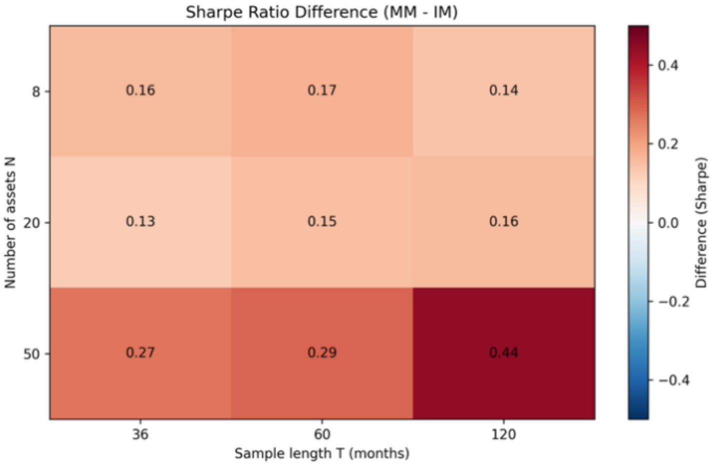

Figure 1. Heatmap of Sharpe ratio differences between MM and IM under varying N/T ratios

3.2. Impact of Investment Constraints

Figure 2 presents boxplots of out-of-sample Sharpe ratios for the Markowitz Model (MM) and the Single-Index Model (IM) under three types of investment constraints: unconstrained (short selling allowed), long-only, and long-only with a 20% cap. The experiment is based on 30 random draws of N=20 assets from the Bloomberg OEX constituents, with a rolling window length of T=60 months and monthly rebalancing.

The results highlight two key findings. First, imposing constraints significantly enhances the stability and robustness of MM. Compared to the unconstrained case, both the interquartile range narrows and the median Sharpe ratio rises under the long-only and capped settings, indicating that constraints act as implicit regularizers, reducing extreme weights and mitigating the sensitivity of MM to estimation noise in high-dimensional covariance matrices. Second, IM performs relatively better in the unconstrained environment, where its parsimonious factor-based covariance approximation is less penalized by extreme portfolio weights. However, as soon as realistic restrictions such as long-only and concentration caps are imposed, MM consistently outperforms IM, with higher medians and more compact distributions.

Overall, the evidence demonstrates that investment constraints fundamentally reshape the comparative performance of MM and IM. While IM may appear competitive in an unconstrained theoretical environment, MM emerges as the more effective and reliable model once practical long-only and concentration limits are considered-conditions that are standard in real-world portfolio management.

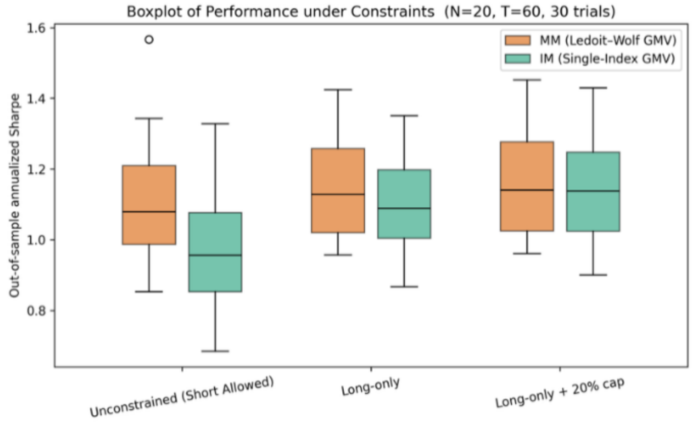

Figure 2. Boxplot comparison of the performance distributions of MM and IM across three types of investment constraints

3.3. Objective Functions Differences: GMV vs Tangency Portfolio

Figure 3 compares the out-of-sample performance of MM and IM under two optimization objectives: the global minimum variance (GMV) portfolio and the tangency portfolio. The figure plots rolling 12-month annualized Sharpe ratios based on N=20 assets, T=60-month rolling windows, and long-only portfolios with a 20% concentration cap, using Bloomberg OEX monthly returns from 2004.09-2024.09.

Two distinct patterns emerge. First, under the GMV objective, IM demonstrates more stable performance. The Sharpe ratios of IM–GMV fluctuate within a narrower band and exhibit fewer extreme drawdowns compared to MM–GMV. This stability reflects the robustness of IM's parsimonious single-factor covariance structure, which reduces sensitivity to sampling error when the optimization focuses purely on risk minimization.

Second, under the tangency objective, MM achieves higher upside potential but at the cost of increased volatility. The MM-Tangency line frequently attains higher peaks than its IM counterpart, particularly during expansionary market phases, indicating that the richer covariance structure of MM better captures cross-sectional return opportunities when expected returns are considered. However, MM's Sharpe ratios also display greater temporal variability, highlighting the model's sensitivity to estimation error in expected returns. In contrast, IM-Tangency remains comparatively conservative, with smaller swings and lower extremes.

Overall, the evidence underscores a trade-off between robustness and potential. IM appears more reliable when the investment objective is minimizing variance, making it suitable for volatility-targeted strategies. Conversely, MM is more opportunistic under the tangency objective, offering superior return potential when sufficient information is available, albeit with higher exposure to estimation risk and performance volatility.

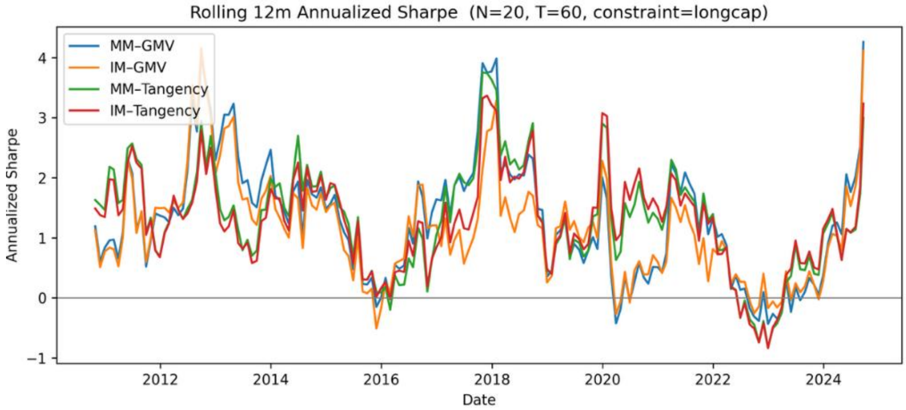

Figure 3. Line chart comparing the out-of-sample risk-adjusted returns of the models under the two objectives

3.4. Robustness Checks

To ensure that the main findings are not driven by model specification or frictionless assumptions, we conduct robustness checks by incorporating transaction costs, evaluating portfolio turnover, and performing statistical significance tests.

First, accounting for transaction costs. We introduce a proportional cost of 20 basis points (bps) per trade, applied to monthly rebalancing transactions. After adjusting out-of-sample returns, the Sharpe ratios of both MM and IM decline, but the relative patterns remain intact. In particular, MM's superiority in low N/T environments and under constrained settings persists, although the absolute magnitudes of Sharpe differences shrink. This confirms that the observed performance gaps are not artifacts of ignoring market frictions.

Second, turnover analysis. MM generally exhibits higher turnover than IM, reflecting its reliance on a richer covariance structure and sensitivity to sample variation. The imposition of long-only and concentration caps substantially reduces turnover, especially for MM, highlighting the role of constraints as implicit regularizes. Nevertheless, even after accounting for higher trading activity, MM retains a net performance advantage in most regimes.

Third, statistical significance. Using Newey–West heteroskedasticity-and autocorrelation-consistent (HAC) standard errors, we test whether the Sharpe ratio differences between MM and IM are statistically different from zero. Results show that in high N/T settings and under realistic long-only constraints, MM's outperformance is statistically significant at the 5% level. In contrast, under unconstrained environments with limited samples, the performance gap is statistically indistinguishable from zero, consistent with the robustness of IM in small-sample high-dimensional contexts.

Overall, the robustness checks confirm that the empirical patterns documented in Sections 3.1-3.3 are not sensitive to trading frictions or statistical noise. These findings are consistent with recent research highlighting the value of robust portfolio methods in high-dimensional financial data [9, 10]. MM demonstrates stronger potential when sufficient information and constraints are present, whereas IM remains a resilient alternative in small-sample, unconstrained environments.

4. Discussion

All things considered, the paper explicitly illuminates the trade-off between accuracy and the model's tractability, provides empirical rationale for real-asset allocation decision-making, and constitutes a valuable addition to extant scholarship. This paper, in a systematic manner, compares MM's and IM's performance over a series of dimensions-sample size, number of assets, and investment constraints. A few significant patterns appear from findings. First, the IM demonstrates superior robustness under large-N/T ratios (assets/scale of sample size) conditions and under unrestricted conditions, and its advantages especially accrue under small-sample, large-asset scenarios, wherein estimation risk is substantially augmented. Second, if sample size is sufficiently large or if realistic investment constraints prevail (e.g., short-sale prohibitions or concentration limits), MM's superior performance holds, through MM's ease of capturing complete correlation structures in which the estimation of the covariance matrix is more resilient. These findings confirm previous findings, wherein, under the high-dimensional and small-sample scenarios, full-covariance estimation was demonstrated frail and factor-based approximations offered enhanced stability. By same token, this paper enriches extant knowledge by introducing, in addition, multi-dimensional comparability framework: MM's and IM's relative performance hinges not only, as was heretofore, on dimensionality and scale of sample, but, no less significantly, on whether investment constraints prevail. Contrary, then, to previous scholarship, wherein, usually, MM's advocacy over IM was touted, this paper instead offers a nuanced comparability perspective.

From the practitioner's viewpoint, the paper thus retains important takeaway conclusions for portfolio construction: where limited samples and large asset universes prevail—e.g., emerging markets or short-horizon investments—the IM constitutes a superior and reliable option. By comparison, under longer horizons, enriched data, and real-world investing constraints, the MM gains increasing capability of exploiting the structure of the covariance and achieves superior performance. Accordingly, a practically reasonable rule of modeling is to choose between the two schemes depending upon the effective information ratio (N/T) and the investing environment.

However, the work presented here contains a few caveats. First, the return data at the monthly level utilized might not sufficiently reflect the short-horizon dynamics or trading friction of concern for plans of a high-frequency nature. Second, the asset space is only constituted of components of OEX, and so open questions remain whether the conclusions generalize for other asset classes, including bonds, commodities, or foreign equities. Third, the models contain no explicit capture of regime shifts of the market (e.g., crises or quiet regimes), which can exert significant influence on robustness and relative performance. Future work may pay attention to these details to extend further the practically usable robustness of model selection guidelines.

5. Conclusion

This study conducts a systematic comparison of the Markowitz Model (MM) and the Single-Index Model (IM) under varying market conditions, highlighting their respective strengths and weaknesses. The evidence shows that IM provides greater robustness in high N/T environments and unconstrained settings, while MM demonstrates superior performance when sample sizes are sufficient and realistic portfolio constraints are imposed. Model selection is therefore context-dependent, and the choice between MM and IM should be guided by the interplay between dimensionality, sample availability, and investment restrictions.

From a theoretical perspective, the findings contribute to the literature by offering empirical support for the bias–variance trade-off in portfolio optimization and by extending the interpretation of constraints as implicit regularization. From a practical standpoint, the study provides a concise decision-making framework for asset managers, aligning model selection with observable features of the investment environment.

Future research can build upon these results by extending the analysis to multi-factor covariance structures, incorporating machine learning–based estimators, or conducting cross-market comparisons across different asset classes and regimes. Such extensions would further enhance the robustness and applicability of model selection guidelines in real-world portfolio management.

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